Hypergeometric Solutions to the q-Painlevé Equation of Type $(A_1 + A_1')^{(1)}$

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Abstract

A class of classical solutions to the q-Painlevé equation of type $(A_1 + A_1')^{(1)}$ (a q-difference analog of the Painlevé II equation) is constructed in a determinantal form with basic hypergeometric function elements. The continuous limit of this q-Painlevé equation to the Painlevé II equation and its hypergeometric solutions are discussed. The continuous limit of these hypergeometric solutions to the Airy function is obtained through a uniform asymptotic expansion of their integral representation.

1 Introduction

In this article we consider the following q-difference equation

$$\left(\overline{F}F - 1\right)(F\underline{F} - 1) = \frac{at^2 F}{F + t}, \tag{1}$$

where t is the independent variable

$$\bar{t} = qt, \quad \underline{t} = t/q, \quad F = F(t), \quad \overline{F} = F(qt), \quad \underline{F} = F(t/q), \tag{2}$$

with $|q| < 1$ and a is a parameter. Equation (1) was identified as one of the q-difference Painlevé equations by Ramani and Grammaticos[26] with a continuous limit to the Painlevé II equation ($\mathrm{P_{II}}$). In this sense eq.(1) is sometimes regarded as a q-analog of $\mathrm{P_{II}}$. Sakai formulated the discrete Painlevé equations as Cremona transformations on a certain family of rational surfaces and developed their classification theory[30]. According to this theory eq.(1) is a discrete dynamical system on the rational surface characterized by the Dynkin diagram of type $A_6^{(1)}$, which posseses the symmetry of the affine Weyl group of type $(A_1 + A_1')^{(1)}$. Equation (1) may be denoted as $\mathrm{dP}(A_6^{(1)})$ by the notation adopted in [20]. Although eq.(1) is the simplest nontrivial q-difference Painlevé equation that admits a Bäcklund transformation only a few results are known - its continuous limit[26] and its simplest hypergeometric solution[28, 8, 9]. The first purpose of this article is to construct "higher-order" hypergeometric solutions to eq.(1) explicitly in determinantal form.

The second purpose of this article is to consider the continuous limit in some detail. The limiting procedure works well on the formal level of the defining q-difference equation however naïve application of the procedure does not work on the level of their solutions. The application of the continuous limit to the series representation of the basic hypergeometric functions that appear in the solutions does not yield the Airy functions which are the hypergeometric solutions of $\mathrm{P_{II}}$. To obtain the valid limit we follow the procedure used by Prellberg[25] - we construct an appropriate integral representation of the function and derive an asymptotic expansion by applying a generalization of the saddle point method.

Our paper is organized as follows. In section 2 we construct hypergeometric solutions to eq.(1). The simplest solution is obtained in section 2.1, and determinant formula of "higher-order" solutions is presented in section 2.2, whose proof is given in section 2.3. In section 3 we consider the continuous limit as $q \to 1^-$. The limit on the formal level is discussed in section 3.1. We discuss the limit on the level of hypergeometric functions in section 3.2. Section 4 is devoted to concluding remarks.

2 Hypergeometric Solutions and Their Determinant Formula

2.1 The Simplest Solution

The simplest hypergeometric solution to eq.(1) is obtained by looking for the special case where it reduces to the Riccati equation. Then by linearizing the Riccati equation we obtain a second order linear q-differential equation, which admits basic hypergeometric functions as solutions.

Let us first recall the definition of the basic hypergeometric series[4]

$$ {}_r\varphi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,z\right) = \sum_{n=0}^{\infty} \frac{(a_1,\ldots,a_r;q)_n}{(b_1,\ldots,b_s;q)_n(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+r-s} z^n, \tag{3}$$

where

$$ (a_1,\ldots,a_r;q)_n = \prod_{i=1}^{n}(a_i;q)_n, \quad (a;q)_n = (1-a)(1-qa)\cdots(1-q^{n-1}a). \tag{4}$$

Then the simplest hypergeometric solution to eq.(1) is given as follows (see also [8, 9, 28]):

Lemma 2.1. *Eq.(1) admits the following particular solution for $a = q$:*

$$ F = \frac{\overline{\psi}}{\psi}, \tag{5}$$

where $\psi(t)$ satisfies the linear q-difference equation

$$ \overline{\psi} + t\psi = \underline{\psi}. \tag{6}$$

The general solution of eq.(6) is given by

$$ \psi(t) = A\ {}_1\varphi_1\left(\begin{matrix} 0 \\ -q \end{matrix};q,-qt\right) + B\ \mathrm{e}^{\pi i \frac{\log t}{\log q}}\ {}_1\varphi_1\left(\begin{matrix} 0 \\ -q \end{matrix};q,qt\right), \tag{7}$$

where A and B are arbitrary q-periodic functions.

Proof. It is easy to see that if F is the solution of the Riccati equation

$$ \overline{F} = \frac{1}{F} - qt, \tag{8}$$

then F satisfies eq.(1) with $a = q$. Equation (8) can be linearized via eq.(5) to eq.(6) by putting $F = \phi/\psi$ and equating the numerators and denominators of both sides.

We next substitute $\psi = t^{\rho}\sum_{n=0}^{\infty} a_n t^n$ into eq.(6). Then we have $q^{2\rho} = 1$, which implies $\rho = \frac{m\pi i}{\log q}$ ($m \in \mathbb{Z}$). Furthermore we deduce a recursion relation for a_n from eq.(6)

$$ a_n = (-1)^m \frac{q^n}{1-q^{2n}} a_{n-1}. $$

Accordingly we obtain two fundamental solutions to eq.(6) as

$$ \psi_1(t) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(q;q)_n(q;-q)_n}(-qt)^n, \quad \psi_2(t) = \mathrm{e}^{\pi i \frac{\log t}{\log q}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(q;q)_n(q;-q)_n}(qt)^n, $$

for $m = 0, 1$ respectively. This completes the proof of Lemma 2.1. □

2.2 Bäcklund Transformation and Determinant Formula

Sakai constructed the following transformations for the homogeneous variables x, y, z and the parameters a_0, a_1, b on the $A_6^{(1)}$ type (*Mul.7*) surface[1]:

$$\begin{aligned}
&\sigma: \quad (a_1,a_0,b\,;\; x:y:z) \mapsto (a_0,a_1,a_1b\,;\; z(z+x):bx(z+x):yz)\\
&\sigma_{(15)(24)(60)}: \quad (a_1,a_0,b\,;\; x:y:z)\\
&\qquad\qquad \mapsto (1/a_0,1/a_1,b\,;\; z(x+z)(x+y+z):y((z+bx)(x+z)+yz):bx(x+z)^2)\\
&w_1: \quad (a_1,a_0,b\,;\; x:y:z) \mapsto (1/a_1,a_1^2a_0,a_1b\,;\; x(y+a_1z):y(y+z):a_1z(y+z))\\
&w_0 = \sigma_{(15)(24)(60)}w_1\sigma_{(15)(24)(60)}.
\end{aligned} \tag{9}$$

The action of σ^2

$$\sigma^2: \quad (a_1,a_0,b\,;\; x:y:z) \longmapsto (a_1,a_0,a_0a_1b\,;\; yz^2(x+y+z):a_1bz^2(x+y+z)(x+z):bxyz(x+z)). \tag{10}$$

gives rise to eq.(1) by putting

$$a = \left(\frac{a_0}{a_1}\right)^{1/2}, \quad t = \left(\frac{b^2}{a_0^3a_1}\right)^{1/4}, \quad q = (a_0a_1)^{1/2}, \qquad F = \left(\frac{a_0}{a_1b^2}\right)^{1/4}\frac{y}{x}. \tag{11}$$

Note that these transformations satisfy the fundamental relations,

$$w_0^2 = w_1^2 = 1, \quad w_0\sigma^2 = \sigma^2w_0, \quad w_1\sigma^2 = \sigma^2w_1.$$

Then w_0 and w_1 can be regarded as Bäcklund transformations of eq.(1). In particular, the action of $T = \sigma w_1$ is given by

$$T(a) = q^2a, \quad T(t) = t, \quad T(F) = \frac{qat\overline{F} + \overline{F}F - 1}{(\overline{F}F - 1)(t\overline{F} + \overline{F}F - 1)}. \tag{12}$$

Therefore applying T to the "seed" solution in Lemma 2.1, we obtain "higher-order" hypergeometric solutions to eq.(1) expressible in terms of a rational function of ψ for $a = q^{2N+1}$ ($N \in \mathbb{Z}$). It is observed that the numerators and denominators of such solutions are factorized and those factors admit the following Casorati determinant formula.

Theorem 2.2. *For each $N \in \mathbb{Z}$, we define $\tau_N(t)$ by*

$$\tau_N(t) = \begin{cases}
\begin{vmatrix} \psi(t) & \psi(q^2t) & \cdots & \psi(q^{2N-2}t)\\ \psi(q^{-1}t) & \psi(qt) & \cdots & \psi(q^{2N-3}t)\\ \vdots & \vdots & & \vdots\\ \psi(q^{-N+1}t) & \psi(q^{-N+3}t) & \cdots & \psi(q^{N-1}t)\end{vmatrix} & (N>0),\\[2ex]
1 & (N=0),\\[2ex]
\begin{vmatrix} \psi(q^{-1}t) & \psi(q^{-3}t) & \cdots & \psi(q^{-2M+1}t)\\ \psi(t) & \psi(q^{-2}t) & \cdots & \psi(q^{-2M+2}t)\\ \vdots & \vdots & & \vdots\\ \psi(q^{M-2}t) & \psi(q^{M-4}t) & \cdots & \psi(q^{-M}t)\end{vmatrix} & (M=-N,\ N<0).
\end{cases} \tag{13}$$

Then

$$F(t) = \begin{cases} \dfrac{1}{q^N}\,\dfrac{\tau_N(t)\tau_{N+1}(qt)}{\tau_N(qt)\tau_{N+1}(t)} & (N \geq 0),\\[2ex] -\dfrac{1}{q^{N+1}}\,\dfrac{\tau_N(t)\tau_{N+1}(qt)}{\tau_N(qt)\tau_{N+1}(t)} & (N<0),\end{cases} \tag{14}$$

satisfies eq.(1) with $a = q^{2N+1}$.

[1] Actions of these transformations are modified from the original formulae in [30] so that they are subtraction-free. Some typographical errors have been also fixed.

Theorem 2.2 is the direct consequence of the following proposition.

Proposition 2.3. *The τ_N satisfy the following bilinear q-difference equations of Hirota type:*

(1) $N \geq 0$

$$q^{2N}\tau_{N+1}(t/q)\tau_N(q^2t) - q^N t\,\tau_{N+1}(t)\tau_N(qt) - \tau_{N+1}(qt)\tau_N(t) = 0, \tag{15}$$

$$q^{2N}\tau_{N+1}(t/q)\tau_N(qt) - q^{2N} t\,\tau_{N+1}(t)\tau_N(t) - \tau_{N+1}(qt)\tau_N(t/q) = 0. \tag{16}$$

(2) $N < 0$

$$q^{2N+2}\tau_{N+1}(t/q)\tau_N(q^2t) + q^{N+1} t\,\tau_{N+1}(t)\tau_N(qt) - \tau_{N+1}(qt)\tau_N(t) = 0, \tag{17}$$

$$q^{2N+2}\tau_{N+1}(t/q)\tau_N(qt) + q^{2N+1} t\,\tau_{N+1}(t)\tau_N(t) - \tau_{N+1}(qt)\tau_N(t/q) = 0. \tag{18}$$

In fact one can derive Theorem 2.2 from Proposition 2.3 as follows: for $N \geq 0$ the bilinear equations eqs.(15) and (16) can be rewritten as

$$q^{2N} - \frac{\mu_{N+1}(t/q)}{\mu_N(qt)}\left(q^N t + \frac{\mu_{N+1}(t)}{\mu_N(t)}\right) = 0, \quad q^{2N} - q^{2N} t\,\frac{\mu_{N+1}(t/q)}{\mu_N(t)} - \frac{\nu_N(qt)}{\nu_N(t/q)} = 0, \tag{19}$$

respectively by introducing the variables

$$\nu_N(t) = \frac{\tau_{N+1}(t)}{\tau_N(t)}, \quad \mu_N(t) = \frac{\tau_N(qt)}{\tau_N(t)}. \tag{20}$$

Putting

$$F = \frac{1}{q^N}\frac{\nu_N(qt)}{\nu_N(t)} = \frac{1}{q^N}\frac{\tau_{N+1}(qt)\tau_N(t)}{\tau_{N+1}(t)\tau_N(qt)}, \tag{21}$$

and eliminating μ_N and μ_{N+1} from eq.(19) through the use of the identity

$$\frac{\nu_N(qt)}{\nu_N(t)} = \frac{\mu_{N+1}(t)}{\mu_N(t)}, \tag{22}$$

we obtain eq.(1) with $a = q^{2N+1}$. The case of $N < 0$ can be verified in the same way.

2.3 Proof of Proposition 2.3

Our basic idea for proving Proposition 2.3 is to use a determinantal technique. Bilinear q-difference equations are derived from the Plücker relations which are quadratic identities among determinants whose columns are shifted. Therefore, we first construct such "difference formulas" that relate "shifted determinants" and τ_N by using the q-difference equation of ψ. We then derive bilinear difference equations with the aid of difference formulas from proper Plücker relations. We refer to [7, 10, 12, 13, 21] for applications of this method to hypergeometric solutions of other discrete Painlevé equations.

Let us consider the case of $N > 0$. We first introduce a notation for the determinants

$$\tau_N(t) = \begin{vmatrix} \psi(t) & \psi(q^2t) & \cdots & \psi(q^{2N-2}t) \\ \psi(q^{-1}t) & \psi(qt) & \cdots & \psi(q^{2N-3}t) \\ \vdots & \vdots & & \vdots \\ \psi(q^{-N+1}t) & \psi(q^{-N+3}t) & \cdots & \psi(q^{N-1}t) \end{vmatrix} = |\,\Psi_0, \Psi_2, \ldots, \Psi_{2N-2}\,|, \tag{23}$$

where Ψ_k denotes a column vector

$$\Psi_k = \begin{pmatrix} \psi(q^k t) \\ \psi(q^{k-1}t) \\ \vdots \\ \psi(q^{k-N+1}t) \end{pmatrix}. \tag{24}$$

Here the height of the column vector is N however we employ the same symbol for determinants with differing heights.

Lemma 2.4. *The following formulas hold:*

$$| \Psi_0, \Psi_2, \ldots, \Psi_{2N-2} | = \tau_N(t), \tag{25}$$

$$| \widehat{\Psi}_0, \Psi_1, \Psi_3, \ldots, \Psi_{2N-3} | = (-1)^{N-1} q^{-(N-1)(N-2)/2} t^{-N+1}\ \tau_N(t), \tag{26}$$

$$| \Psi_1, \widehat{\Psi}_2, \Psi_3, \ldots, \Psi_{2N-3} | = (-1)^{N-2} q^{-(N-1)(N-2)/2} t^{-N+1}\ \tau_N(t), \tag{27}$$

where $\widehat{\Psi}_k$ *denotes the column vector*

$$\widehat{\Psi}_k = \begin{pmatrix} \psi(q^k t) \\ q\psi(q^{k-1}t) \\ \vdots \\ q^{N-1}\psi(q^{k-N+1}t) \end{pmatrix}. \tag{28}$$

Proof. Using the linear equation eq.(6) for ψ on the N-th column of the determinant eq.(23) we find

$$\begin{aligned} \tau_N(t) &= \begin{vmatrix} \psi(t) & \psi(q^2t) & \cdots & \psi(q^{2N-4}t) & \psi(q^{2N-4}t) - q^{2N-3}t\psi(q^{2N-3}t) \\ \psi(q^{-1}t) & \psi(qt) & \cdots & \psi(q^{2N-5}t) & \psi(q^{2N-5}t) - q^{2N-4}t\psi(q^{2N-4}t) \\ \vdots & \vdots & & \vdots & \\ \psi(q^{-N+1}t) & \psi(q^{-N+3}t) & \cdots & \psi(q^{N-3}t) & \psi(q^{N-3}t) - q^{N-2}t\psi(q^{N-2}t) \end{vmatrix}, \\ &= \begin{vmatrix} \psi(t) & \psi(q^2t) & \cdots & -q^{2N-3}t\psi(q^{2N-3}t) \\ \psi(q^{-1}t) & \psi(qt) & \cdots & -q^{2N-4}t\psi(q^{2N-4}t) \\ \vdots & \vdots & & \vdots \\ \psi(q^{-N+1}t) & \psi(q^{-N+3}t) & \cdots & -q^{N-2}t\psi(q^{N-2}t) \end{vmatrix}. \end{aligned}$$

Applying the same procedure from the $(N-1)$-th column to the second column we have

$$\begin{aligned} \tau_N(t) &= \begin{vmatrix} \psi(t) & -qt\psi(qt) & \ldots & -q^{2N-3}t\psi(q^{2N-3}t) \\ \psi(q^{-1}t) & -t\psi(t) & \ldots & -q^{2N-4}t\psi(q^{2N-4}t) \\ \vdots & \vdots & & \vdots \\ \psi(q^{-N+1}) & -q^{-N+2}t\psi(q^{-N+2}t) & \ldots & -q^{N-2}t\psi(q^{N-2}t) \end{vmatrix} \\ &= (-t)^{N-1} q^{(N-1)^2} \begin{vmatrix} \psi(t) & \psi(qt) & \cdots & \psi(q^{2N-3}t) \\ \psi(q^{-1}t) & q^{-1}\psi(t) & \cdots & q^{-1}\psi(q^{2N-4}t) \\ \vdots & \vdots & & \vdots \\ \psi(q^{-N+1}t) & q^{-N+1}\psi(q^{-N+2}t) & \cdots & q^{-N+1}\psi(q^{N-2}t) \end{vmatrix} \\ &= (-t)^{N-1} q^{(N-1)(N-2)/2} \begin{vmatrix} \psi(t) & \psi(qt) & \cdots & \psi(q^{2N-3}t) \\ q\psi(q^{-1}t) & \psi(t) & \cdots & \psi(q^{2N-4}t) \\ \vdots & \vdots & & \vdots \\ q^{N-1}\psi(q^{-N+1}t) & \psi(q^{-N+2}t) & \cdots & \psi(q^{N-2}t) \end{vmatrix} \\ &= (-t)^{N-1} q^{(N-1)(N-2)/2}\ | \widehat{\Psi}_0, \Psi_1, \Psi_3, \ldots, \Psi_{2N-3} |, \end{aligned}$$

which is nothing but eq.(26). At the stage where the above procedure has been employed up to the third column we have

$$\tau_N(t) = \begin{vmatrix} \psi(t) & \psi(q^2t) & -q^3t\psi(q^3t) & \ldots & -q^{2N-3}t\psi(q^{2N-3}t) \\ \psi(q^{-1}t) & \psi(qt) & -q^2t\psi(q^2t) & \ldots & -q^{2N-4}t\psi(q^{2N-4}t) \\ \vdots & \vdots & \vdots & & \vdots \\ \psi(q^{-N+1}t) & \psi(q^{-N+3}t) & -q^{-N+4}t\psi(q^{-N+4}t) & \ldots & -q^{N-2}t\psi(q^{N-2}t) \end{vmatrix}.$$

Using eq.(6) on the first column, we obtain

$$
\begin{aligned}
\tau_N(t) &= \begin{vmatrix} \psi(q^2t)+qt\psi(qt) & \psi(q^2t) & -q^3t\psi(q^3t) & \dots & -q^{2N-2}\psi(q^{2N-3}t) \\ \psi(qt)+t\psi(t) & \psi(qt) & -q^2t\psi(q^2t) & \dots & -q^{2N-3}\psi(q^{2N-4}t) \\ \vdots & \vdots & \vdots & & \vdots \\ \psi(q^{-N+3}t)+q^{-N+2}t\psi(q^{-N+2}t) & \psi(q^{-N+3}t) & -q^{-N+4}t\psi(q^{-N+4}t) & \dots & -q^{N-1}\psi(q^{N-2}t) \end{vmatrix} \\
&= (-1)^{N-2}q^{(N-1)(N-2)/2}t^{N-1} \begin{vmatrix} \psi(qt) & \psi(q^2t) & \psi(q^3t) & \cdots & \psi(q^{2N-3}t) \\ \psi(t) & q\psi(qt) & \psi(q^2t) & \dots & \psi(q^{2N-4}t) \\ \vdots & \vdots & \vdots & & \vdots \\ \psi(q^{-N+2}t) & q^{N-3}\psi(q^{-N+3}t) & \psi(q^{-N+4}t) & \dots & \psi(q^{N-2}t) \end{vmatrix} \\
&= (-1)^{N-2}q^{(N-1)(N-2)/2}t^{N-1} \mid \Psi_1, \widehat{\Psi}_2, \Psi_3, \dots, \Psi_{2N-3} \mid,
\end{aligned}
$$

which is eq. (27). □

Now consider the Plücker relation,

$$
\begin{aligned}
0 &= \left| \Psi_{-1}, \widehat{\Psi}_0, \Psi_1, \dots, \Psi_{2N-5} \right| \times \mid \Psi_1, \Psi_3, \dots, \Psi_{2N-3}, \phi \mid \\
&- \left| \widehat{\Psi}_0, \Psi_1, \Psi_3, \dots, \Psi_{2N-5}, \phi \right| \times \mid \Psi_{-1}, \Psi_1, \Psi_3, \dots, \Psi_{2N-3} \mid \\
&+ \mid \Psi_{-1}, \Psi_1, \Psi_3, \dots, \Psi_{2N-5}, \phi \mid \times \left| \widehat{\Psi}_0, \Psi_1, \Psi_3, \dots, \Psi_{2N-3} \right|,
\end{aligned}
$$

for an arbitrary column vector ϕ. In particular by choosing ϕ as

$$
\phi = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 1 \end{pmatrix},
$$

and applying Lemma 2.4 we obtain eq.(15) from the former and eq.(16) from the latter, respectively. The case of $N < 0$ can be proved in a similar manner. This completes the proof of Proposition 2.3. □

3 Continuous Limit to $\mathrm{P_{II}}$

3.1 Continuous Limit on the Formal Level

The continuous limit of "q-$\mathrm{P_{II}}$" to $\mathrm{P_{II}}$ involves the "quantum" to classical limit $q \to 1$ but in contrast to the trivial limits usually employed in basic hypergeometric series, i.e. making the substitution $z \mapsto (1-q)z$ and then setting $q \to 1$, we have a completely different limiting process which is far from trivial and has not been studied much. Let us first recall the formal limits of $\mathrm{dP}(A_6^{(1)})$ eq.(1).

Proposition 3.1. *[26] With the replacements*

$$
F = i\mathrm{e}^{-\delta w}, \quad a = \mathrm{e}^{-\frac{\eta}{2}\delta^3}, \quad q = \mathrm{e}^{-\frac{\delta^3}{2}}, \quad t = -2i\mathrm{e}^{-\frac{s}{2}\delta^2} = -2iq^{\frac{s}{\delta}}, \tag{29}
$$

eq.(1) has a limit to $\mathrm{P_{II}}$

$$
\frac{d^2w}{ds^2} = 2w^3 + 2sw + \eta, \tag{30}
$$

as $\delta \to 0$.

Proposition 3.1 can be easily verified by noticing that

$$
w(q^{\pm 1}t) = w(s \pm \delta) = w \pm \delta \frac{d}{ds}w + \frac{\delta^2}{2!}\frac{d^2w}{ds^2} + O(\delta^3). \tag{31}
$$

It is well-known that P_{II} eq.(30) admits the hypergeometric solutions for $\eta = 2N+1$ ($N \in \mathbb{Z}$)[24]:

$$w = -\frac{d}{ds}\log\frac{\kappa_{N+1}}{\kappa_N}, \tag{32}$$

$$\kappa_N = \begin{cases} \begin{vmatrix} v & \dfrac{dv}{ds} & \cdots & \left(\dfrac{d}{ds}\right)^{N-1} v \\ \dfrac{dv}{ds} & \dfrac{d^2v}{ds^2} & \cdots & \left(\dfrac{d}{ds}\right)^{N} v \\ \vdots & \vdots & \ddots & \vdots \\ \left(\dfrac{d}{ds}\right)^{N-1} v & \left(\dfrac{d}{ds}\right)^{N} v & \cdots & \left(\dfrac{d}{ds}\right)^{2N-2} v \end{vmatrix} & (N>0), \\ 1 & (N=0), \\ \begin{vmatrix} v & \dfrac{dv}{ds} & \cdots & \left(\dfrac{d}{ds}\right)^{M-1} v \\ \dfrac{dv}{ds} & \dfrac{d^2v}{ds^2} & \cdots & \left(\dfrac{d}{ds}\right)^{M} v \\ \vdots & \vdots & \ddots & \vdots \\ \left(\dfrac{d}{ds}\right)^{M-1} v & \left(\dfrac{d}{ds}\right)^{M} v & \cdots & \left(\dfrac{d}{ds}\right)^{2M-2} v \end{vmatrix} & (N=-M<0), \end{cases} \tag{33}$$

where v satisfies the Airy equation,

$$\frac{d^2v}{ds^2} = -sv. \tag{34}$$

The general solution to eq.(34) is given by

$$v(s) = C\mathrm{Ai}(\mathrm{e}^{\frac{\pi i}{3}}s) + D\mathrm{Ai}(\mathrm{e}^{-\frac{\pi i}{3}}s), \tag{35}$$

where C, D are arbitrary constants and $\mathrm{Ai}(s)$ is the Airy function defined by

$$\mathrm{Ai}(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\frac{i}{3}u^3+isu}\,du. \tag{36}$$

Proposition 3.2. *Under the substitutions eq.(29) and with*

$$\psi(t) = \mathrm{e}^{\frac{\pi i}{2}\frac{\log t}{\log q}}v(t), \tag{37}$$

the hypergeometric solutions to eq.(1) for $a = q^{2N+1}$ ($N \in \mathbb{Z}$), eqs.(6), (13) and (14) yield, in the limit $\delta \to 0$, the hypergeometric solutions to P_{II} eq.(30) for $\eta = 2N+1$, eqs. (34), (33) and (32).

Proof. Noticing that

$$\psi(q^{\pm 1}t) = \pm i\mathrm{e}^{\frac{\pi i}{2}\frac{\log t}{\log q}}v(s\pm\delta),$$

the linear equation eq.(6) can then be rewritten as

$$v(s+\delta) + v(s-\delta) = 2\mathrm{e}^{-\frac{s}{2}\delta^2}v(s), \tag{38}$$

which yields eq.(34) in the limit $\delta \to 0$. We next consider the limit of τ_N and the dependent variable transformation.

The determinant τ_N $(N > 0)$ can be rewritten as

$$\begin{aligned}
\tau_N &= e^{\frac{N\pi i}{2}\frac{\log t}{\log q}} \begin{vmatrix} v(t) & i^2 v(q^2 t) & \cdots & i^{2N-2} v(q^{2N-2} t) \\ i^{-1} v(q^{-1} t) & i v(qt) & \cdots & i^{2N-3} v(q^{2N-3} t) \\ \vdots & \vdots & & \vdots \\ i^{-N+1} v(q^{-N+1} t) & i^{-N+3} v(q^{-N+3} t) & \cdots & i^{N-1} v(q^{N-1} t) \end{vmatrix} \\
&= e^{\frac{N\pi i}{2}\frac{\log t}{\log q}} i^{N(N-1)/2} \begin{vmatrix} v(s) & v(s+2\delta) & \cdots & v(s+(2N-2)\delta) \\ v(s-\delta) & v(s+\delta) & \cdots & v(s+(2N-3)\delta) \\ \vdots & \vdots & & \vdots \\ v(s-(N-1)\delta) & v(s-(N-3)\delta) & \cdots & v(s+(N-1)\delta) \end{vmatrix} \\
&= e^{\frac{N\pi i}{2}\frac{\log t}{\log q}} \, i^{N(N-1)/2} \, \sigma_N(s),
\end{aligned}$$

so that

$$F = \frac{i}{q^N} \frac{\sigma_N(s)\sigma_{N+1}(s+\delta)}{\sigma_N(s+\delta)\sigma_{N+1}(s)}. \tag{39}$$

We also note that

$$\sigma_N(s) = (-2\delta^2)^{N(N-1)/2} \; [\kappa_N + O(\delta)] \, .$$

Therefore we deduce

$$w = -\frac{1}{\delta}\log\frac{F}{i} = -\frac{1}{\delta}\log\left[1 + \delta\left(\frac{d}{ds}\log\frac{\kappa_{N+1}}{\kappa_N}\right) + O(\delta^2)\right] = -\frac{d}{ds}\log\frac{\kappa_{N+1}}{\kappa_N} + O(\delta). \tag{40}$$

The limit in the case of $N < 0$ can be verified in a similar manner. □

Now let us consider the limit of the solution of the linear equation eq.(6). The two linearly independent power-series solutions of the Airy equation eq.(34) are given by

$$\begin{aligned}
{}_0F_1\left(\begin{matrix} - \\ \frac{2}{3} \end{matrix}; -\frac{s^3}{3^2}\right) &= 1 - \frac{1}{3!}s^3 + \frac{1\cdot 4}{6!}s^6 - \frac{1\cdot 4\cdot 7}{9!}s^9 + \cdots, \\
s \; {}_0F_1\left(\begin{matrix} - \\ \frac{4}{3} \end{matrix}; -\frac{s^3}{3^2}\right) &= s - \frac{2}{4!}s^4 + \frac{2\cdot 5}{7!}s^7 - \frac{2\cdot 5\cdot 8}{10!}s^{10} + \cdots .
\end{aligned} \tag{41}$$

However as is apparent from the series expansion of hypergeometric functions in eq.(7) the application of the scaling changes of variables in eq.(29) does not yield any meaningful limit as $\delta \to 0$ on a term by term basis. What is required is another representation of these functions and a uniform, possibly asymptotic, expansion with respect to the other parameters as $q \to 1$. This question has been addressed in [25] and for the most part answered there. We discuss the continous limit of the hypergeometric functions ${}_1\varphi_1\left(\begin{matrix} 0 \\ -q \end{matrix}; q, \mp qt\right)$ in the next section.

3.2 Continuous Limit of the Hypergeometric Functions

There are three key ingredients in [25] which are necessary to derive the final formula that we require. The first is a suitable integral representation for the ${}_1\varphi_1\left(\begin{matrix} 0 \\ y \end{matrix}; q, x\right)$ function and this is the q-analog of the Mellin-Barnes inversion integral.

Proposition 3.3. *[4, 25] The representation*

$${}_1\varphi_1\left(\begin{matrix} 0 \\ y \end{matrix}; q, x\right) = \frac{(q;q)_\infty}{(y;q)_\infty}\int_{\rho-i\infty}^{\rho+i\infty} \frac{dz}{2\pi i}\, z^{-\log x/\log q}\, \frac{(y/z;q)_\infty}{(z;q)_\infty}, \tag{42}$$

is valid for $x, y \in \mathbb{C}$, with $|\arg(x)| < \pi$, $y \neq q^{-n}$ ($n \in \mathbb{Z}_{\geq 0}$), $0 < \rho < 1$ and $0 < q < 1$.

Remark 3.1. Proposition 3.3 follows by evaluating the integral on the contour described in Fig.1 with the residues at $z = q^{-n}$

$$\operatorname*{Res}_{z=q^{-n}} (z;q)_\infty^{-1} = (-1)^{n+1}\frac{q^{\binom{n}{2}}}{(q;q)_n (q;q)_\infty}, \quad n \in \mathbb{Z}_{\geq 0}, \tag{43}$$

and by deforming the path C appropriately according to Cauchy's Theorem.

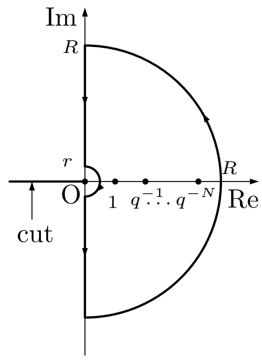

Figure 1: Contour for Proposition 3.3.

The second ingredient is an asymptotic formula for the q-shifted factorial $(t;q)_\infty$ as $q \to 1$ which is uniform with respect to t. Such expansions have only recently been studied and in particular by Meinardus [19], McIntosh [16, 17, 18] and the above cited work [25]. Amongst all the essentially equivalent forms we choose the following statement:

Proposition 3.4. *[19, 25, 18] As $q \to 1^-$ the q-shifted factorial $(t;q)_\infty$ has an asymptotic expansion*

$$\log(t;q)_\infty \sim \frac{1}{\log q}\mathrm{Li}_2(t) + \frac{1}{2}\log(1-t) + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}\left(t\frac{d}{dt}\right)^{2n-2} \frac{t}{1-t}(\log q)^{2n-1}, \tag{44}$$

for $0 < q < 1$ and uniform for t in any compact domain of $\mathbb{C}$ such that $|\arg(1-t)| < \pi$. Here $\mathrm{Li}_2(t)$ is the dilogarithm function defined by

$$\mathrm{Li}_2(t) = -\int_0^t \frac{\log(1-u)}{u}\,du = \sum_{n=1}^{\infty} \frac{t^n}{n^2}, \tag{45}$$

and B_{2n} the even Bernoulli numbers. In the case of $t = q$ we have [22]

$$\log(q;q)_\infty = \frac{\pi^2}{6\log q} + \frac{1}{2}\log\frac{2\pi}{-\log q} + O(\log q). \tag{46}$$

We next apply Proposition 3.4 to the integral representation eq.(42). Noticing that q-shifted factorials in the integral representation are rewritten by putting $q = \mathrm{e}^{-\epsilon}$ as

$$\frac{(y/z;q)_\infty}{(z;q)_\infty} z^{-\log x/\log q} = \mathrm{e}^{\frac{1}{\epsilon}\left[-\mathrm{Li}_2(y/z)+\mathrm{Li}_2(z)+\log x \log z\right]} \times \mathrm{e}^{\frac{1}{2}\left[\log\left(1-\frac{y}{z}\right)-\log(1-z)\right]} \times [1+O(\epsilon)],$$

and

$$\frac{(q;q)_\infty}{(y;q)_\infty} = \mathrm{e}^{\frac{1}{\epsilon}\left[\mathrm{Li}_2(y)-\frac{\pi^2}{6}\right]+\frac{1}{2}\log\frac{2\pi}{\epsilon}-\frac{1}{2}\log(1-y)} \times [1+O(\epsilon)],$$

we obtain:

Proposition 3.5. *Let $x, y \in \mathbb{C}$ and $q = \mathrm{e}^{-\epsilon}$ for $\epsilon > 0$. Then*

$$ {}_1\varphi_1\left(\begin{matrix}0\\ y\end{matrix}; q, x\right) = \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} \mathrm{e}^{\frac{1}{\epsilon}\left[\log x \log z - \mathrm{Li}_2(\frac{y}{z}) + \mathrm{Li}_2(z)\right]} \mathrm{e}^{\frac{1}{2}\left[\log(1-\frac{y}{z}) - \log(1-z)\right]}\,dz \times [1+O(\epsilon)]\,, \tag{47}$$

where $|\arg(x)| < \pi$, $\mathrm{Re}\, y < \rho$.

Remark 3.2. We take $\log z$ on its principal sheet cut along $(-\infty, 0]$ and $\mathrm{Li}_2(z)$ on its principal sheet cut along $(1, +\infty)$. If $x, y \in (0,1)$ then for $z \in (y, 1)$ the argument of

$$\log x \log z - \mathrm{Li}_2\left(\frac{y}{z}\right) + \mathrm{Li}_2(z) \tag{48}$$

is zero. When $z \in \mathbb{C}$ subject to

$$|\arg z| < \pi, \quad |\arg(1-z)| < \pi, \quad \left|\arg\left(1-\frac{y}{z}\right)\right| < \pi,$$

that is we exclude the rays $(1,\infty)$, $(-\infty,0)$ and $(0,y)$, it follows that the argument of eq.(48) lies in the interval $(-\pi,\pi)$. The contour path given in Proposition 3.5 is then just a simple path $z = \rho + it$ ($\mathrm{Re}\, y < \rho < 1$, $t \in (-\infty,\infty)$) satisfying these criteria and the requirement that the endpoints of the contour ensure the existence of the integral. If the contour is deformed then eq.(47) is valid if the contour does not cut across the ray $(0,y)$.

The third ingredient is the application of saddle point method to the Laplace type integral in eq.(47). In this problem two saddle points arise and can coalesce depending on the values of the parameters. Therefore we have to construct an asymptotic approximation that incorporates the contributions from both saddle points uniformly with repsect to their separation. A method for such an asymptotic expansion of this type of integral has been set out by Chester, Friedman and Ursell[2, 32, 33].

To illustrate the method, let us consider the integral

$$I = \frac{1}{2\pi i}\int_C \mathrm{e}^{\frac{1}{\epsilon}g(z;d)} f(z)\, dz, \quad \epsilon \to 0,$$

where $f(z)$ is analytic with respect to z, and $g(z;d)$ is analytic with respect to z and the parameter d. We assume that there are two saddle points z_1 and z_2 which are determined from $g'(z;d) = 0$ and that they coalesce when $d = 0$. The key of this method is to introduce the change of variable $z \to u$ via the cubic parameterization

$$g(z;d) = \frac{1}{3}u^3 - \alpha u + \beta, \tag{49}$$

where α and β are determined as follows. Firstly differentiating eq.(49) we have

$$g'(z;d)\frac{dz}{du} = u^2 - \alpha. \tag{50}$$

In order for eq.(49) to define a single-valued analytic transformation neither $\frac{dz}{du}$ nor $\frac{du}{dz}$ can vanish in relevant regions. Therefore at the saddle points we have the correspondence

$$z = z_1 \leftrightarrow u = \alpha^{\frac{1}{2}}, \quad z = z_2 \leftrightarrow u = -\alpha^{\frac{1}{2}}, \tag{51}$$

which determines α and β as

$$\frac{4}{3}\alpha^{\frac{2}{3}} = g(z_2;d) - g(z_1;d), \quad 2\beta = g(z_2;d) + g(z_1;d). \tag{52}$$

The transformation $u = u(z;d)$ defined by eqs.(49) and (52) has three branches. However, it can be shown that there is exactly one branch which has the following properties[2, 33]; (i) $u = u(z;\alpha)$ is expanded into a power series in z with coefficient continuous in d near $d = 0$, (ii) z_1 and z_2 correspond to $\alpha^{\frac{1}{2}}$ and $-\alpha^{\frac{1}{2}}$ respectively, and (iii) near $d = 0$ the correspondence $z \leftrightarrow u$ is 1:1.

We next expand $f(z)$ in the form

$$f(z)\frac{dz}{du} = \sum_{m=0}^{\infty}(p_m + q_m u)(u^2-\alpha)^m, \tag{53}$$

and define the following integrals

$$\begin{aligned} F_m &= \frac{1}{2\pi i}\int_{C'} \mathrm{e}^{\frac{1}{\epsilon}\left(\frac{1}{3}u^3-\alpha u\right)}(u^2-\alpha)^m\, du, \\ G_m &= \frac{1}{2\pi i}\int_{C'} \mathrm{e}^{\frac{1}{\epsilon}\left(\frac{1}{3}u^3-\alpha u\right)}(u^2-\alpha)^m u\, du, \end{aligned} \qquad m = 0,1,\ldots. \tag{54}$$

Here C' is the image of C by the transformation given by eqs.(49) and (52). By using recursion relations for F_m and G_m obtained by partial integration, the expansion of I can be written in the form

$$I = \mathrm{e}^{\frac{\beta}{\epsilon}}\left[\epsilon^{\frac{1}{3}}V(\alpha\epsilon^{\frac{2}{3}})\sum_{m=0}^{\infty}a_m\epsilon^m + \epsilon^{\frac{2}{3}}V'(\alpha\epsilon^{\frac{2}{3}})\sum_{m=0}^{\infty}b_m\epsilon^m\right], \tag{55}$$

where

$$a_0 = p_0, \quad b_0 = -q_0, \tag{56}$$

and $V(\lambda)$ is the Airy integral

$$V(\lambda) = \frac{1}{2\pi i}\int_{C'}\mathrm{e}^{\frac{1}{3}u^3-\lambda u}du. \tag{57}$$

The coefficients p_0 and q_0 are determined by putting $z = z_1$, $u = \alpha^{\frac{1}{2}}$ and $z = z_2$, $u = -\alpha^{\frac{1}{2}}$ in eq.(53) as

$$\begin{aligned} p_0 &= \frac{1}{2}\left[f(z_1)\left(\frac{dz}{du}\right)_{z=z_1} + f(z_2)\left(\frac{dz}{du}\right)_{z=z_2}\right],\\ q_0 &= \frac{1}{2\alpha^{\frac{1}{2}}}\left[f(z_1)\left(\frac{dz}{du}\right)_{z=z_1} - f(z_2)\left(\frac{dz}{du}\right)_{z=z_2}\right]. \end{aligned} \tag{58}$$

Prellberg [25] has applied the above expansion to the integral representation eq.(47) to obtain its leading behaviour as $\epsilon \to 0$ for $0 < x, y < 1$ as follows:

Proposition 3.6. *[25] Let* $0 < x, y < 1$ *and* $q = e^{-\epsilon}$ *with* $\epsilon > 0$. *Then as* $\epsilon \to 0$ *we have*

$$\begin{aligned} {}_1\varphi_1\left(\begin{matrix}0\\ y\end{matrix}; q, x\right) &= \mathrm{e}^{\frac{1}{\epsilon}[\mathrm{Li}_2(y)-\frac{1}{6}\pi^2+\frac{1}{2}\log x\log y]}\sqrt{\frac{2\pi}{\epsilon(1-y)}}\\ &\quad\times\left[p_0\epsilon^{\frac{1}{3}}\mathrm{Ai}(\alpha\epsilon^{-\frac{2}{3}}) - q_0\epsilon^{\frac{2}{3}}\mathrm{Ai}'(\alpha\epsilon^{-\frac{2}{3}})\right][1+\mathrm{O}(\epsilon)], \end{aligned} \tag{59}$$

where the auxiliary variables are

$$\frac{4}{3}\alpha^{3/2} = \log x\log\left(\frac{z_m-\sqrt{d}}{z_m+\sqrt{d}}\right) + 2\mathrm{Li}_2(z_m-\sqrt{d}) - 2\mathrm{Li}_2(z_m+\sqrt{d}), \tag{60}$$

with

$$z_m = \frac{1+y-x}{2}, \qquad d = z_m^2 - y, \tag{61}$$

and

$$p_0 = \left(\frac{\alpha}{d}\right)^{1/4}\frac{1-x-y}{2}, \qquad q_0 = \left(\frac{d}{\alpha}\right)^{1/4}. \tag{62}$$

Remark 3.3. Errors in the statement of this result as given in [25] have been fixed, in particular the sign of the q_0 term in eq.(59) and the factor of 2 in the denominator of p_0 in eq.(62).

In our case, the parameters are given by

$$\epsilon = \frac{\delta^3}{2}, \quad q = \mathrm{e}^{-\frac{\delta^3}{2}}, \quad y = -\mathrm{e}^{-\frac{\delta^3}{2}} = -1 + \frac{\delta^3}{2}, \quad x = \mp qt = \mp 2i\mathrm{e}^{-\frac{s}{2}\delta^2+\frac{\delta^3}{2}} = \pm(2i - is\delta^2 - i\delta^3 + \cdots), \tag{63}$$

and unfortunately, they do not match the assumptions of Proposition 3.6. Therefore we have to consider the extension of Proposition 3.6 for our case, taking care of the multi-valuedness of the integrand. We fix the branch of log and fractional power functions as

$$\log z = \ln|z| + i\mathrm{Arg}\, z, \quad -\pi \le \mathrm{Arg}\, z < \pi, \qquad z^{\frac{m}{n}} = \exp\left(\frac{m}{n}\log z\right) = |z|^{\frac{m}{n}}\mathrm{e}^{\frac{m}{n}i\mathrm{Arg}\, z}.$$

Note that $\log(XY) = \log X + \log Y$ and $\log(X/Y) = \log X - \log Y$ are valid only mod $2\pi i$.

Substituting eq.(63) into eq.(47) and expanding the integrand with respect to δ, we obtain:

Proposition 3.7. *With the substitutions eq.(63) we have as $\delta \to 0$*

$$
{}_1\varphi_1\left(\begin{matrix}0\\-q\end{matrix}\,;\,q,-qt\right) = \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} \mathrm{e}^{\frac{2}{\delta^3}g_-(z)} f_-(z)\,dz \times \left(\frac{2\pi}{\delta^3}\right)^{\frac12} \mathrm{e}^{\frac{2}{\delta^3}\left(-\frac{\pi^2}{4}+\frac{\ln 2}{2}\delta^3\right)} \times \left[1+O(\delta^3)\right], \tag{64}
$$

$$
{}_1\varphi_1\left(\begin{matrix}0\\-q\end{matrix}\,;\,q,qt\right) = \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} \mathrm{e}^{\frac{2}{\delta^3}g_+(z)} f_+(z)\,dz \times \left(\frac{2\pi}{\delta^3}\right)^{\frac12} \mathrm{e}^{\frac{2}{\delta^3}\left(-\frac{\pi^2}{4}+\frac{\ln 2}{2}\delta^3\right)} \times \left[1+O(\delta^3)\right], \tag{65}
$$

where $0<\rho<1$ *and*

$$
g_-(z) = \mathrm{Li}_2(z) - \mathrm{Li}_2\left(-\frac{1}{z}\right) + \log\left(2i\mathrm{e}^{-\frac{s\delta^2}{2}}\right)\log z, \tag{66}
$$

$$
f_-(z) = \mathrm{e}^{-\frac12\log\left(1+\frac1z\right)-\frac12\log(1-z)-\log z}, \tag{67}
$$

$$
g_+(z) = \mathrm{Li}_2(z) - \mathrm{Li}_2\left(-\frac{1}{z}\right) + \log\left(-2i\mathrm{e}^{-\frac{s\delta^2}{2}}\right)\log z, \tag{68}
$$

$$
f_+(z) = \mathrm{e}^{-\frac12\log\left(1+\frac1z\right)-\frac12\log(1-z)-\log z}. \tag{69}
$$

Let us take the case of $x=-qt$ and apply the saddle point method to eqs.(64), (66) and (67). The saddle points $z_1^{(-)}$ and $z_2^{(-)}$ are determined by

$$
g_-'(z) = -\frac{\log(1-z)}{z} - \frac{\log\left(1+\frac1z\right)}{z} + \frac{\log\left(2i\mathrm{e}^{-\frac{s\delta^2}{2}}\right)}{z} = 0, \tag{70}
$$

which yields the quadratic equation

$$
z^2 + 2i\mathrm{e}^{-\frac{s\delta^2}{2}}z - 1 = 0. \tag{71}
$$

Therefore the saddle points are given by

$$
z_1^{(-)} = z_m + D, \quad z_2^{(-)} = z_m - D, \quad z_m = -i\mathrm{e}^{-\frac{s\delta^2}{2}}, \quad D^2 = z_m^2 + 1, \tag{72}
$$

or expanding in terms of δ we obtain

$$
z_m = -i + \frac{is}{2}\delta^2 + \cdots, \quad D = s^{\frac12}\delta - \frac{s^{\frac32}}{4}\delta^3 + \cdots, \tag{73}
$$

$$
z_1^{(-)} = -i + s^{\frac12}\delta + \frac{is}{2}\delta^2 + \cdots, \quad z_2^{(-)} = -i - s^{\frac12}\delta + \frac{is}{2}\delta^2 + \cdots. \tag{74}
$$

The quantity α is calculated by using eq.(52) as

$$
\begin{aligned}
\frac43\alpha^{\frac32} &= g(z_2^{(-)}) - g(z_1^{(-)}) \\
&= \mathrm{Li}_2\left(z_m - D\right) - \mathrm{Li}_2\left(-\frac{1}{z_m-D}\right) + \log\left(2i\mathrm{e}^{-\frac{s\delta^2}{2}}\right)\log(z_m-D) \\
&\quad -\mathrm{Li}_2\left(z_m + D\right) + \mathrm{Li}_2\left(-\frac{1}{z_m+D}\right) - \log\left(2i\mathrm{e}^{-\frac{s\delta^2}{2}}\right)\log(z_m+D) \\
&= 2\left[\mathrm{Li}_2\left(z_m-D\right) - \mathrm{Li}_2\left(z_m+D\right)\right] + \log\left(2i\mathrm{e}^{-\frac{s\delta^2}{2}}\right)\left[\log(z_m-D) - \log(z_m+D)\right] \\
&= \frac{2i}{3}s^{\frac32}\delta^3 + O(\delta^5)
\end{aligned}
$$

where we have used $\frac{1}{z_m\pm D} = -(z_m\mp D)$. Therefore we conclude

$$
\alpha = 2^{-\frac23}s\mathrm{e}^{\frac{\pi i}{3}}\delta^2 + O(\delta^4). \tag{75}
$$

We can derive this for s in the sector $-\pi \leq \mathrm{Arg}(s) < \pi/3$ but it actually holds without this restriction. The reason for this is that ${}_1\varphi_1$ is an analytic function of $t \in \mathbb{C}$ and therefore of $s \in \mathbb{C}$. Consequently the leading term of the expansion of ${}_1\varphi_1$ as $\delta \to 0$ is analytic with respect to s as the remainder terms can be shown to be uniformly bounded in s under this limit. Let us next compute p_0 and q_0 according to the formula eq.(58). From the correspondence

$$
\begin{aligned}
z = z_1^{(-)} = -i + s^{\frac{1}{2}}\delta + \frac{is}{2}\delta^2 + \cdots \quad &\longleftrightarrow \quad u = u_1^{(-)} = \alpha^{\frac{1}{2}} = 2^{-\frac{1}{3}} s^{\frac{1}{2}} \mathrm{e}^{\frac{\pi i}{6}} \delta + O(\delta^3) \\
z = z_2^{(-)} = -i - s^{\frac{1}{2}}\delta + \frac{is}{2}\delta^2 + \cdots \quad &\longleftrightarrow \quad u = u_2^{(-)} = -\alpha^{\frac{1}{2}} = -2^{-\frac{1}{3}} s^{\frac{1}{2}} \mathrm{e}^{\frac{\pi i}{6}} \delta + O(\delta^3)
\end{aligned}
$$

we obtain $(dz/du)_{z=z_1^{(-)},z_2^{(-)}}$ as

$$
\begin{aligned}
\left(\frac{dz}{du}\right)_{z=z_1^{(-)}} &= \frac{\left(\frac{dz_1}{d\delta}\right)_{\delta\sim 0}}{\left(\frac{du_1^{(-)}}{d\delta}\right)_{\delta\sim 0}} = \frac{s^{\frac{1}{2}} + is\delta + O(\delta^2)}{2^{-\frac{1}{3}} s^{\frac{1}{2}} \mathrm{e}^{\frac{\pi i}{6}} + O(\delta^2)} = 2^{\frac{1}{3}} \mathrm{e}^{-\frac{\pi i}{6}} \left(1 + is^{\frac{1}{2}}\delta + O(\delta^2)\right), \\
\left(\frac{dz}{du}\right)_{z=z_2^{(-)}} &= \frac{\left(\frac{dz_2}{d\delta}\right)_{\delta\sim 0}}{\left(\frac{du_2^{(-)}}{d\delta}\right)_{\delta\sim 0}} = \frac{-s^{\frac{1}{2}} + is\delta + O(\delta^2)}{-2^{-\frac{1}{3}} s^{\frac{1}{2}} \mathrm{e}^{\frac{\pi i}{6}} + O(\delta^2)} = 2^{\frac{1}{3}} \mathrm{e}^{-\frac{\pi i}{6}} \left(1 - is^{\frac{1}{2}}\delta + O(\delta^2)\right).
\end{aligned}
$$

Substituting eq.(74) into eq.(67) we have

$$
f(z_1^{(-)}) = 2^{-\frac{1}{2}} \mathrm{e}^{\frac{\pi i}{4}} \left(1 - is^{\frac{1}{2}}\delta + \cdots\right), \quad f(z_2^{(-)}) = 2^{-\frac{1}{2}} \mathrm{e}^{\frac{\pi i}{4}} \left(1 + is^{\frac{1}{2}}\delta + \cdots\right),
$$

from which we obtain

$$
p_0 = \frac{1}{2}\left[f(z_1^{(-)}) \left(\frac{dz}{du}\right)_{z=z_1^{(-)}} + f(z_2^{(-)}) \left(\frac{dz}{du}\right)_{z=z_2^{(-)}} \right] = 2^{-\frac{1}{6}} \mathrm{e}^{\frac{\pi i}{12}} + O(\delta^2), \tag{76}
$$

$$
q_0 = \frac{1}{2\alpha^{\frac{1}{2}}}\left[f(z_1^{(-)}) \left(\frac{dz}{du}\right)_{z=z_1^{(-)}} - f(z_2^{(-)}) \left(\frac{dz}{du}\right)_{z=z_2^{(-)}} \right] = 0 + O(\delta). \tag{77}
$$

We compute β by using eq.(52) as

$$
\begin{aligned}
2\beta &= g(z_2^{(-)}) + g(z_1^{(-)}) \\
&= \mathrm{Li}_2\,(z_m - D) - \mathrm{Li}_2\left(-\frac{1}{z_m - D}\right) + \log\left(2i\mathrm{e}^{-\frac{s\delta^2}{2}}\right)\log(z_m - D) \\
&+ \mathrm{Li}_2\,(z_m + D) - \mathrm{Li}_2\left(-\frac{1}{z_m + D}\right) + \log\left(2i\mathrm{e}^{-\frac{s\delta^2}{2}}\right)\log(z_m + D) \\
&= \log\left(2i\mathrm{e}^{-\frac{s\delta^2}{2}}\right)\left[\log(z_m - D) + \log(z_m + D)\right],
\end{aligned} \tag{78}
$$

which yields

$$
\beta = -\frac{\pi i}{2}\ln 2 + \frac{\pi^2}{4} + \frac{\pi i}{4} s\delta^2 + O(\delta^4). \tag{79}
$$

Remark 3.4. The multi-valuedness of the integrand has a critical effect in the calculation of β. One might compute β from eq.(78) as

$$
\begin{aligned}
2\beta &= \log\left(2i\mathrm{e}^{-\frac{s\delta^2}{2}}\right)\left[\log(z_m - D) + \log(z_m + D)\right] = \log\left(2i\mathrm{e}^{-\frac{s\delta^2}{2}}\right)\log(z_m^2 - D^2) \\
&= \log\left(2i\mathrm{e}^{-\frac{s\delta^2}{2}}\right)\log(-1) = -\pi i(\ln 2 + \frac{\pi i}{2} - \frac{s\delta^2}{2}),
\end{aligned}
$$

but the second equality does not hold in general (in this case it is accidentally correct). In fact, the same procedure for the case of $x = qt$ yields wrong result.

Let us finally consider the image of integration path $C : z = \rho + it$ $(-\infty < t < \infty)$ in the u-plane. From the identity of dilogarithm[14, 3]

$$\mathrm{Li}_2(z) = -\mathrm{Li}_2(\frac{1}{z}) - \frac{1}{2}(\log z)^2 + \pi i \log z + \frac{\pi^2}{3}, \tag{80}$$

we see for $t \to \pm\infty$

$$\begin{aligned} g_-(\rho + it) \quad &\sim \quad \mathrm{Li}_2(\rho + it) = -\mathrm{Li}_2(\frac{1}{\rho + it}) - \frac{1}{2}(\log(\rho + it))^2 + \pi i \log(\rho + it) + \frac{\pi^2}{3} \\ &\sim \quad -\frac{1}{2}(\log|t|)^2 \pm \frac{\pi i}{2}\log|t|, \quad t \to \pm\infty, \end{aligned} \tag{81}$$

therefore

$$u(\rho + it) \sim (3g_-(\rho + it))^{\frac{1}{3}} \sim \mathrm{e}^{\pm\frac{\pi i}{3}}\left(\frac{3}{2}(\log|t|)^2\right)^{\frac{1}{3}}, \quad t \to \pm\infty.$$

This gives the integration path C' as $\infty\mathrm{e}^{-\frac{\pi i}{3}} \to O \to \infty\mathrm{e}^{\frac{\pi i}{3}}$, which implies $V(\lambda) = \mathrm{Ai}(\lambda)$. Closer investigation shows that the mapping given by eq.(49) is regular and 1:1 in the domain including C'.

Collecting the above results and performing similar calculations for the case of $x = qt$, we obtain the following asymptotic expansions from eq.(55):

Proposition 3.8. *With the substitutions eq.(29) we have as* $\delta \to 0$

$$\begin{aligned} {}_1\varphi_1\left(\begin{matrix} 0 \\ -q \end{matrix} ;\ q, -qt\right) &= 2\pi^{\frac{1}{2}}\delta^{-\frac{1}{2}}\mathrm{e}^{-\frac{\pi i}{\delta^3}\ln 2 + \frac{\pi i}{2\delta}s + \frac{\pi i}{12}}\left[\mathrm{Ai}(s\mathrm{e}^{\frac{\pi i}{3}}) + O(\delta^2)\right], \qquad (82) \\ {}_1\varphi_1\left(\begin{matrix} 0 \\ -q \end{matrix} ;\ q, qt\right) &= 2\pi^{\frac{1}{2}}\delta^{-\frac{1}{2}}\mathrm{e}^{\frac{\pi i}{\delta^3}\ln 2 - \frac{\pi i}{2\delta}s - \frac{\pi i}{12}}\left[\mathrm{Ai}(s\mathrm{e}^{-\frac{\pi i}{3}}) + O(\delta^2)\right], \qquad (83) \end{aligned}$$

for s in any compact domain of $\mathbb{C}$.

Now we are in a position to deduce the limit of the general solution to eq.(6). From eqs.(37) and (7) we note that

$$v(t) = A\,\mathrm{e}^{-\frac{\pi i}{2}\frac{\log t}{\log q}}\,{}_1\varphi_1\left(\begin{matrix} 0 \\ -q \end{matrix} ; q, -qt\right) + B\,\mathrm{e}^{\frac{\pi i}{2}\frac{\log t}{\log q}}\,{}_1\varphi_1\left(\begin{matrix} 0 \\ -q \end{matrix} ; q, qt\right), \tag{84}$$

for arbitrary q-periodic functions A, B. Observing that

$$\mathrm{e}^{\pm\frac{\pi i}{2}\frac{\log t}{\log q}} = \mathrm{e}^{\frac{1}{\delta^3}\left(\mp\pi i \ln 2 \mp \frac{\pi^2}{2}\right) \pm \frac{\pi i}{2\delta}s} \times [1 + O(\delta)], \tag{85}$$

we find that the s-dependence in the exponential pre-factors of eqs.(82) and (83) cancels exactly. Therefore we finally arrive at the desired result:

Theorem 3.9. *We have as* $\delta \to 0$

$$v(t) = 2\sqrt{\frac{\pi}{\delta}}\left[A\,\mathrm{e}^{\frac{\pi^2}{2\delta^3} + \frac{\pi i}{12}}\mathrm{Ai}(\mathrm{e}^{\frac{\pi i}{3}}s) + B\,\mathrm{e}^{-\frac{\pi^2}{2\delta^3} - \frac{\pi i}{12}}\mathrm{Ai}(\mathrm{e}^{-\frac{\pi i}{3}}s)\right] \times [1 + O(\delta)], \tag{86}$$

for s in any compact domain of $\mathbb{C}$ and constants $A, B \in \mathbb{C}$.

4 Concluding Remarks

In this article we have considered the q-Painlevé equation $\mathrm{dP}(A_6^{(1)})$, eq.(1), and constructed its classical solutions having a determinantal form with basic hypergeometric function elements. We have also discussed the continuous limit to $\mathrm{P_{II}}$. In particular, we have shown that hypergeometric functions ${}_1\varphi_1\left(\begin{matrix} 0 \\ -q \end{matrix} ; q, \mp qt\right)$ actually reduce to the Airy functions $\mathrm{Ai}(\mathrm{e}^{\pm\frac{\pi i}{3}})$ by applying a generalization of the saddle point method to their integral representations.

We first remark that $\mathrm{P_{II}}$ eq.(30) admits rational solutions for $\eta = 2N$ ($N \in \mathbb{Z}$) which can be expressed in terms of a specialization of 2-core Schur functions[11]. Such solutions are obtained by applying Bäcklund transformations to the simple rational solution that is fixed by the Dynkin diagram automorphism. One would expect similar rational solutions for eq.(1), however Masuda has shown that there is no rational solution fixed by the corresponding Dynkin diagram automorphism [15]. This implies that it is not appropriate to regard eq.(1) simply as a "q-analog of $\mathrm{P_{II}}$".

Secondly, we observe an asymmetry in the structure of determinant formula eq.(13); the shifts in t of the entries are different between the horizontal and the vertical directions. It might be natural to regard this structure as originating from the asymmetry of the root lattice. Actually in other cases where this situation arises, such as the $A_5^{(1)}$ surface("q-$\mathrm{P_{IV}}$" and "q-$\mathrm{P_{III}}$", $(A_2+A_1)^{(1)}$-symmetry)[10, 7], $A_4^{(1)}$ surface("q-$\mathrm{P_V}$", $A_4^{(1)}$-symmetry)[5] or $A_3^{(1)}$ surface("q-$\mathrm{P_{VI}}$", $D_5^{(1)}$-symmetry)[31], the determinant structure is symmetric. Such an asymmetric structure of the determinant is known for several discrete Painlevé equations - one example is the "standard" discrete Painlevé II equation[27, 13]

$$x_{n+1} + x_{n-1} = \frac{(an+b)x_n + c}{1 - x_n^2}, \tag{87}$$

where a, b, c are parameters. Equation (87) may be regarded as a special case of the "asymmetric" discrete Painlevé II equation[2][29, 23, 28]

$$x_{n+1} + x_{n-1} = \frac{(an+b)x_n + c + d(-1)^n}{1 - x_n^2}, \tag{88}$$

when $d = 0$, which is actually a discrete Painlevé equation associated with the $D_5^{(1)}$ surface and arises as a Bäcklund transformation of the Painlevé V equation ($\mathrm{P_V}$). Therefore the hypergeometric solutions for (88) are expressible in terms of the Whittaker function and are the same as those for $\mathrm{P_V}$[28]. However, the hypergeometric solutions for eq.(87) are quite different; the relevant hypergeometric function is the parabolic cylinder function and the determinant structure has the same asymmetry as that for $\mathrm{dP}(A_6^{(1)})$ eq.(1). A similar structure is known for the "standard" discrete (q-)Painlevé III equation ($\mathrm{dP_{III}}$)[27, 12]

$$\frac{x_{n+1}x_{n-1}}{d_1 d_2} = \frac{(x_n - c_1 q^n)(x_n - c_2 q^n)}{(x_n - d_1)(x_n - d_2)}, \tag{89}$$

and the q-Painlevé VI equation (q-$\mathrm{P_{VI}}$) [6, 29],

$$\begin{aligned} y_n y_{n+1} &= \frac{a_3 a_4 (z_{n+1} - b_1 q^n)(z_{n+1} - b_2 q^n)}{(z_{n+1} - b_3)(z_{n+1} - b_4)}, \\ z_n z_{n+1} &= \frac{b_3 b_4 (y_n - a_1 q^n)(y_n - a_2 q^n)}{(y_n - a_3)(y_n - a_4)}, \end{aligned} \qquad \frac{b_1 b_2}{b_3 b_4} = q\frac{a_1 a_2}{a_3 a_4}, \tag{90}$$

where a_i, b_i ($i = 1, 2, 3, 4$), c_j and d_j ($j = 1, 2$) are parameters. The hypergeometric solutions for eq.(90) are given by the basic hypergeometric series ${}_2\varphi_1$ [31] while those for eq.(89) are given by Jackson's q-Bessel function[12].

Thus the results of our study sheds some light on the "degenerated" equations such as eqs.(87) or (89). They are not just special cases of the original "generic" equations. Our results imply that putting $d = 0$ in eq.(88) is not just killing the "parity", but causes qualitative change of the root lattice which in turn results in different hypergeometric solutions and an asymmetry of the determinant formula. Therefore they should be studied independently of the "generic" discrete Painlevé equations in the Sakai's classification[30].

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